Distance in Graphs: Two Old and One New Problems

Abstract

The distance between two vertices in a graph is the length of the shortest path connecting the vertices. Determining the distance between a pair of vertices in a graph is known to be solvable in polynomial time [Dijkstra (1959), Floyd and Warshall (1962)]. Since then, many distance-related problems have arisen through the years.

I will present three problems on which I am currently working. In **degree/diameter problem** we maximize the order of a graph given its maximum degree and diameter. By **metric dimension** the position of a vertex in a graph is uniquely determined using distances. And with **distance-magic labeling** we generalize the well-known magic labeling notion through distances.

The *diameter* of a graph G is the greatest distance between two vertices in G. Considering three parameters in a graph: order, degree, and diameter, we can optimize one parameter while fixing the other two parameters. Among three optimization possibilities, maximizing the order of the graph given its maximum degree and diameter is one problem that has attracted many researchers since first introduced by Moore, Hoffman, and Singleton in 1960. A natural upper bound for the maximum order is known due to Moore and a graph whose order attaining such a bound is then called a Moore graph. Unfortunately, or maybe fortunately for the sake of further research, there only exist very few Moore graphs. We have contributed several results: proving non-existence of graphs of maximum degree 4 with order close to the Moore bound and thus improving the upper bound for graphs with that particular degree; and determining "good" bounds for more restricted versions of the degree/diameter problem (i.e., graphs embedable in fixed surfaces and bipartite Cayley graphs).

Motivated by two real world applications: assigning computer memory network and robotic navigation, the notion of metric dimension was introduced separately by Slater (1975) and Harary and Melter (1976). They called a set of vertices S *resolves* a graph G if every vertex is uniquely determined by its vector of distances to the vertices in S. The *metric dimension* of G, $dim(G)$, is the minimum cardinality of a resolving set of G. Determining the metric dimension of an arbitrary graph has been proved to be NP-hard [Garey and Johnson (1979)] and so research in this area are then constrained towards: characterizing graphs with particular metric dimensions, determining metric dimensions of particular graphs, and constructing algorithm that best approximate metric dimensions. Our research has contributed results in the first two topics.

Vilfred (1994) and Miller, Rodger, Simanjuntak (2003) defined *distance-magic labeling* of a graph G as labeling of vertices in G by consecutive integers from 1 up to the order of G such that the summation of all labels of a vertex's neighbors (i.e., vertices of distance one) is independent of the choice of vertex. In a way, this labeling can be viewed as a natural extension of previously known graph labelings: the magic labeling [Sedlacek (1963), Kotzig and Rosa (1970)] and radio labeling (which is distance-based) [Griggs and Yeh (1992)]. Recently, O'Neal and Slater (2012) has generalized the notion of distance-magic labeling by considering the summation for labels of all vertices with particular distances to the vertex under consideration, not only its neighbors.

A short historical account, known techniques, recent results, and open questions of the three afore-mentioned problems will be presented.